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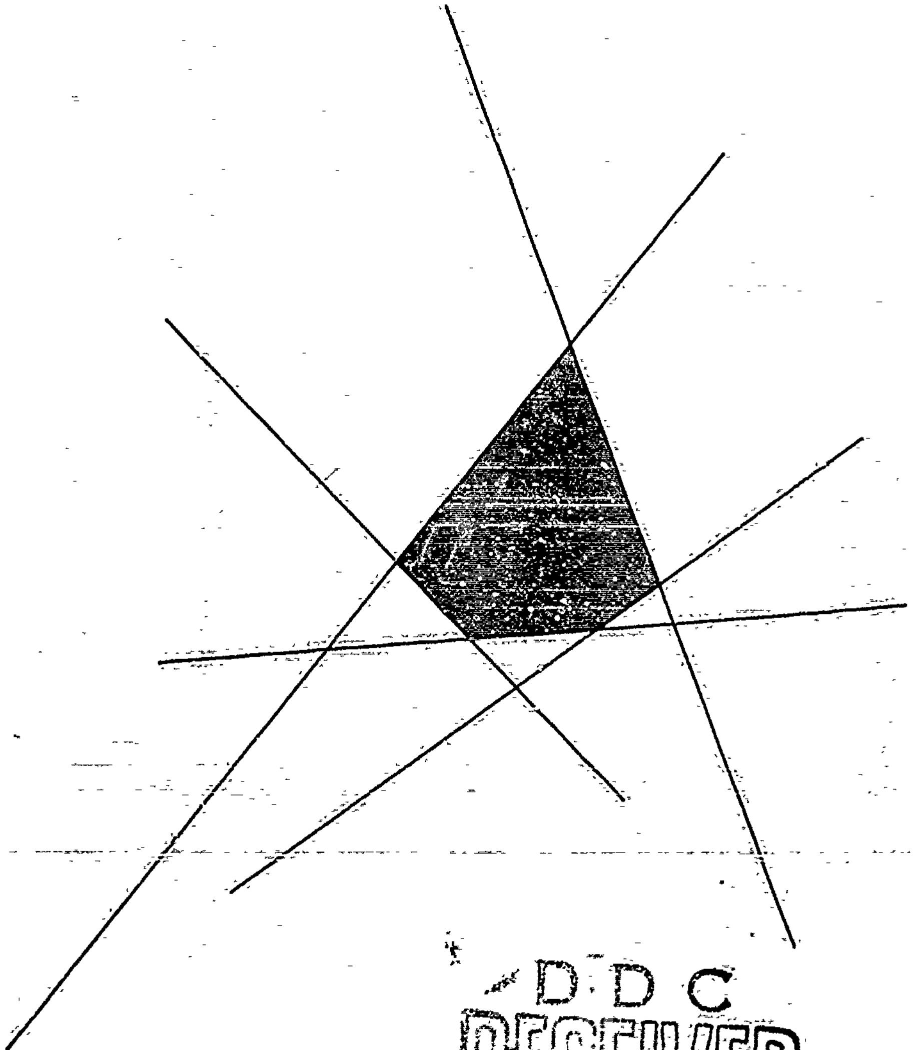
OPPORTUNISTIC REPLACEMENT POLICIES FOR MAINTAINED SYSTEMS

by
DAVINDER P. S. SETHI



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OPPORTUNISTIC REPLACEMENT POLICIES FOR MAINTAINED SYSTEMS[+]

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Davinder P. S. Sethi

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ABSTRACT

Consider a unit in continuous operation. When this unit fails, it is immediately replaced. In addition, opportunities arise according to a renewal process when we can either replace this unit at a reduced cost or do nothing. The problem is formulated as a Markov decision process. If the unit has *increasing failure rate*, the replacement policy that minimizes the expected total discounted cost or the average cost of maintenance is characterized by a single parameter i^* : If an opportunity exists, we replace the working unit only if its age exceeds i^*. Techniques to compute the minimum discounted cost and the optimal policy are suggested.

Under this simple replacement policy structure, the operating characteristics of the system are discussed for the special case where the opportunities arise according to an exponential arrival process.

A series system of two units where the failure of either unit is also an opportunity to replace the other unit at a reduced cost is considered. When the units have *increasing failure rate*, the structure of the optimal policy is again determined.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

The interest and the literature on decision models for maintained systems continues to grow. These models share a common objective of reducing the incidence of system failure or the cost of maintenance over a certain time. However, differing assumptions on the life distribution of the items, the repair facilities and costs, and the available information on the system have led to different models and solution techniques.

Opportunistic models describe situations where it costs less to replace (or repair) two or more units concurrently than to replace them at different times. Thus the necessary replacement of a unit upon its failure may also justify the replacement of some other units whose failure seems imminent. These cost advantages are often due to reduced overhead costs in joint replacements and the economies of scale. Typically, the replacement of an item upon failure is viewed as an opportunity to replace other nonfailed items in the system at a reduced cost. Such a situation may arise when a system is sent to the repair shop for parts replacement. Upon the failure of some critical component, a large cost is incurred in transporting the system to the repair shop and due to services lost while the system waits for repair. Along with the necessary replacements, one may now choose to replace several other items at their marginal cost of replacement. These replacements are called *opportunistic* as opposed to the necessary *failure*

replacements. The criterion for deciding whether an item should be opportunistically replaced or not is called an opportunistic policy. These replacement policies and the advantages in instituting them are largely the subject of this thesis.

First, we review some of the better known replacement models and the situations they describe.

1.2 Replacement Models

In replacement models that seek to reduce maintenance costs, we look for two things. First, the replacement of a unit before failure must cost less than a failure replacement. Secondly, the unit of interest must deteriorate in service. One measure of deterioration with age is called the increasing failure rate (IFR). If the unit has failure distribution F with density function f, then the failure rate function $r(t) = f(t)/1 - F(t)$.

In the area of preventive maintenance, the best known results are for the planned age replacement models (see Barlow and Proschan [1965]). Consider a unit whose replacement upon failure costs C_1 and whose replacement before failure costs $C_2 < C_1$. When the failure distribution F is continuous, the policy that minimizes the expected cost per unit time for an infinite horizon problem is nonrandom: There exists some $t^* \varepsilon [0,\infty]$ such that we replace the unit either upon failure or at age t^* whichever comes first. When the failure rate is continuous and strictly increasing, the optimal replacement age t^* is finite and, is the unique solution to

$$r(t)\int_0^t (1 - F(x))dx - F(t) = \frac{C_2}{C_1 - C_2} .$$

In addition to this model for a single unit system, Barlow and Hunter [1960] introduced an age replacement policy for a complex system sustained with minimal repair. The system is replaced or overhauled at age T at a cost C_1. Intervening failures are rectified at cost C_2 through minimal repair which does not alter the failure rate of the system. When the system failure rate is continuous and strictly increasing, the optimal replacement age T^* is again finite and unique. As before, it may be determined by minimizing the long-run expected cost per unit time over T.

Various other extensions to the basic model have been made. Fox [1966] showed that the age replacement policy also minimizes the total expected discounted cost of maintenance. Schaefer [1971] and, Wolfe and Subramanian [1974], introduced an age dependent cost in the model to reflect the decrease in efficiency of the system with age.

When maintenance costs over a finite time period are of concern, a sequential replacement policy is always better. In this case, every successive replacement age is different and is chosen to minimize the expected costs over the time still remaining. Barlow and Proschan prove that the optimal sequence of replacement times is nonrandom and illustrate its calculation in [1962] and [1965].

For ease of implementation, the block replacement policy has also received attention. Under this policy items of a given type are replaced at failure and at times kT $(k = 1,2, \ldots)$. Let C_1 and C_2 be the costs of a failure replacement and a planned replacement respectively. If $N(T)$ is the expected number of failures in time T, the expected cost per unit time for an infinite horizon problem may be expressed as

$$\frac{C_1 N(T) + C_2}{T} .$$

Minimizing this over T determines the optimal replacement interval. Barlow and Proschan derive this result and, compare it to the age replacement policy in [1965] and [1975].

The periodic and sequential replacement policies referred hereto describe single component systems. These policies extend to multi-component systems if we assume stochastic and economic independence between the components. In such a case, the optimal policy for the system is effected by optimally maintaining each component. One generalization of this model would be a recognition of some form of economic dependence between the components. For instance, opportunistic policies assume there are economies of scale in undertaking joint replacements. Because of the complexity of these models in general, only a few special situations have been analyzed.

Radner and Jorgenson (see [1962] and [1963]) consider a series system of two components, designated 0 and 1. Unit 0 has an increasing failure rate. Suppose it costs C_0 to replace it upon

extensions by Klein, ([1962]), give the decision rules that minimize the cost of maintenance.

Rosenfield considers the above model, ([1974]), with the added complexity of uncertainty about the state of the system. He assumes that at any period we may replace the unit, or inspect it, or do nothing. In every instance an expected operating cost is incurred. Besides, a replacement or an inspection cost is associated with the first two actions. Optimal decision policies that minimize the total discounted cost of maintenance are deduced.

Finally, a recent survey of maintenance models by Pierskalla and Voelker [1975] is mentioned as an exhaustive source of references and related research efforts.

1.3 Thesis Plan

In Chapter 2, we develop an opportunistic replacement model for a single unit. When this unit fails it is immediately replaced. In addition, opportunities arise according to a renewal sequence when we can replace this unit at a reduced cost or do nothing. The opportunities could be the visits of the repairman or the shut-downs of the system of which this unit is a critical component. The system is modeled as a Markov Decision Process, and the optimal policy structure for the average cost and the total discounted cost criterions is established when the unit is IFR. This model also adapts itself to seek marginal improvements in complex systems. However, the case of a series system of two units is explicitly considered. Here, we assume that the failure of either unit is an opportunity to replace the other unit at a reduced cost.

When the units are IFR, the structure of the optimal policy is again determined.

Chapter 3 considers the opportunistic replacements of a unit where the opportunities arise according to a Poisson process. For this special case, the exact solution of the optimal policy is determined. The operating characteristics of this model are studied in detail. In addition, various classes of distributions for Unit 1 and their interaction with an opportunistic policy are considered.

CHAPTER 2

OPTIMAL OPPORTUNISTIC POLICIES

2.1 Introduction

This chapter establishes the structure of the optimal opportunistic replacement policy for two specific cases. In either case, techniques to estimate the minimum cost objective and the parameters of the policy are suggested. The applications and extensions of these basic models are stressed by way of examples.

First, in Section 2, we consider a single unit in continuous operation. When this unit fails it is immediately replaced. In addition, opportunities arise according to a renewal sequence at which times we can replace this unit at a reduced cost or do nothing. When the unit has increasing failure rate, the replacement policy that minimizes the cost of maintaining the unit is characterized by a single parameter i^* : If an opportunity exists, we replace the working unit only if its age exceeds i^*.

In Section 3 we consider a critical system of two units where the failure of either unit is an opportunity to replace the other unit at a reduced cost. When the units have increasing failure rates, the optimal replacement policy is characterized by two parameters, i^* and j^* : If Unit 1 (or Unit 2) fails, we replace both units only if the age of the working unit exceeds i^* (or j^*). In any case, the failed unit is immediately replaced.

All the replacement models treated in this chapter are modeled as Markov Decision Processes and solved explicitly in discrete

time space. The results for the continuous distributions follow immediately. A review of the basic methodology follows. An extensive treatment of this subject can be found in (Derman [1970]) and (Ross [1970]).

Let random vector X_t be the state of the system at time $t = 0,1,2, \ldots$. We observe $X_t = \underline{i}$, and choose an action a_t from a finite set of choices. In doing so, we incur a cost $C(\underline{i},a_t)$ and our next state $X_{t+1} = \underline{j}$ with probability $P_{\underline{i}\underline{j}}(a_t)$. Let π be the policy or rule for choosing the actions. We wish to determine a π that minimizes the "cost of maintaining the system" One measure of our objective is the total expected discounted cost V . Given initial state $\underline{i}$, and policy π we define

$$(1) \quad V_\pi(\underline{i}) = E_\pi \left\{ C(X_0,a_0) + \alpha C(X_1,a_1) + \alpha^2 C(X_2,a_2) + \cdots \mid X_0 = \underline{i} \right\}$$

where α is the rate at which future costs are discounted. For $\alpha < 1$ and $C(\, , \,) < \infty$, $V_\pi(\underline{i})$ is finite and a meaningful objective. Let

$$V(\underline{i}) = \operatorname*{Min}_{\pi} V_\pi(\underline{i}) \; .$$

Under our assumptions, an optimal policy that is stationary (i.e. depends only on the state of the system) exists (for proof see Ross [1970], Theorem 6.3). Let it be π^* . Under policy π^* , $V(\cdot)$ satisfies the functional equation

$$(2) \qquad V(\underline{i}) = \underset{a}{\text{Min}} \left\{ C(\underline{i},a) + \alpha \sum_{\underline{j}} P_{\underline{i}\underline{j}}(a)V(\underline{j}) \right\} \qquad \text{for all } \underline{i} \, .$$

Another objective of interest is the expected average cost of maintaining the system, $\phi(\cdot)$. Given intial state $\underline{i}$, and policy π, we define

$$\phi_\pi(\underline{i}) = \underset{n\to\infty}{\text{Lim}} \, E_\pi \frac{\left\{ \sum_{t=0}^{n} C(X_t,a_t) \mid X_0 = \underline{i} \right\}}{(n+1)} \, .$$

We say policy π^* is optimal if

$$\phi_{\pi^*}(\underline{i}) = \underset{\pi}{\text{Min}} \, \phi_\pi(\underline{i}) \qquad \text{for all } \underline{i} \, .$$

In the replacement models to follow, we minimize the two cost objectives defined above and deduce the nature of the optimal policy π^*.

2.2 Opportunistic Replacements of a Single Unit

2.2.1 The Model

We motivate the formulation of the problem by two examples.

Example (1):

Consider a unit that is in continuous operation. When this unit fails, an emergency crew is called to replace it immediately at a cost $(K + C)$. In addition, a repairman visits the facility periodically and offers to replace the (working) unit at a reduced cost C. If we choose to replace the unit, we save K dollars

but sacrifice the remaining life of the unit in operation. Clearly, every visit of the repairman is a decision state where we may replace the working unit or forego the chance.

Suppose the unit has a discrete failure probability density g_i , $i = 0,1,2, \ldots$. The repairman's visits are also random with probability density f_i , $i = 0,1,2, \ldots$ and $f_0 < 1$. Suppose that replacements, whether opportunistic or upon failure, take 1 unit of time. First, we minimize the total expected discounted cost V_π of maintaining the unit. If future costs are discounted at rate $\alpha < 1$ and $(K + C) < \infty$, then V_π is finite. This follows from the upper bound

$$V_\pi \leq (K + C) + \alpha(K + C) + \alpha^2(K + C) + \cdots \quad \text{for any policy } \pi$$
$$< \infty$$

which is obvious if we note that $(K + C)$ is the maximum we spend in any period.

Example (2):

The above problem is equivalently defined by the following system. Unit 1 is in series with sub-system 2. When 1 fails we replace it at cost $(K + C)$ where K is the cost of system failure. Failure epochs of 2 constitute a renewal sequence and are potential opportunities to replace 1 at reduced cost C , and, are equivalent to the repairman's visits in Example (1). As before, let the failure densities of 1 and 2 be g_i and f_i . Suppose replacements of 1 or 2 or both take one unit of time.

V_π is now the cost associated with maintaining Unit 1 under some replacement policy π.

For convenience, we shall follow Example (2) for further developments. Define the system state by a pair

(i,j) where i : age of Unit 1

j : is the time left to the failure of sub-system 2.

Note that the option to replace 1 or not arises only when sub-system 2 fails, i.e. $j = 0$. So $(i,0)$ is the only decision state. When $j \neq 0$, we do not know its value, nor do we need to know it since (i,j), $j \neq 0$, is not a decision state.

Let R_i be the conditional probability of failure at age i of Unit 1 given survival to age $i - 1$. R_i is the discrete analog of the failure rate (see Barlow and Proschan [1965]) and,

$$R_i = \frac{g_i}{\sum_{k \geq i} g_k} .$$

If $V(i,j)$ is the minimum cost objective given we are in state (i,j), it obeys the functional equations

$$\text{(4a)} \quad \begin{aligned} V(i,j) &= R_i \{K + C + \alpha V(0,j-1)\} + \\ &(1 - R_i) \cdot \alpha V(i+1, j-1) \quad \text{for } j > 0 \end{aligned}$$

and,

$$\text{(4b)} \quad \begin{aligned} V(i,0) &= R_i \left\{C + \alpha \sum_j f_j V(0,j)\right\} + (1 - R_i) \cdot \\ &\text{Min} \left\{C + \alpha \sum_j f_j V(0,j) ; \alpha \sum_j f_j V(i+1,j)\right\} . \end{aligned}$$

When we are in state (i,j) , $j \neq 0$, with probability R_i Unit 1 will fail and the state of the system at the next time period will be $(0,j-1)$. This would entail an immediate expense of $(K+C)$ for system breakdown and replacement plus the discounted value of expected future costs which is $\alpha V(0,j-1)$. In this we assume that we continue to maintain the system using the best available strategy. This accounts for the first term in Equation (4a). Similarly, the second term of Equation (4a) is the survival probability of Unit 1 multiplied by the discounted value of future costs if Unit 1 does not fail. Equation (4b) refers to the decision state. When $j = 0$ (i.e. sub-system 2 fails) and Unit 1 fails concurrently, we take the opportunity at hand and spend only C to replace Unit 1. Our next state is $(0,j)$ with probability f_j . So, the total expected cost is $\left\{C + \alpha \sum_j f_j V(0,j)\right\}$. This accounts for the first term in (4b). If 1 does not fail, we either replace it and expect to spend $C + \alpha \sum_j f_j V(0,j)$, or, do not replace it and expect to spend $\alpha \sum_j f_j (i+1,j)$ depending on which is less.

We now turn our attention to evaluating the optimal objective $V(i,j)$, and, the structure of the associated optimal policy π^* . If Unit 1 has increasing failure rate $(R_i \uparrow i)$, we show that $V(i,j)$ is monotonic in i , and π^* has the following simple structure: There exists $i^* \in (0,\infty]$ such that whenever we are in a decision state $(i,0)$, replace Unit 1 only if $i \geq i^*$.

2.2.2 Computation of the Cost Objective

An explicit solution of the functional equations for $V(i,j)$ is near impossible. However, an interation technique for approaching

functional equations is available. Let $V_0(i,j) = 0$ for all i, j and, define successive approximations by

(5a)
$$V_{k+1}(i,j) = R_i\{K + C + \alpha V_k(0,j-1)\} + (1 - R_i)\cdot \alpha V_k(i+1,j-1) \quad \text{for } j > 0$$

and

(5b)
$$V_{k+1}(i,0) = R_i\left\{C + \alpha \sum_j f_j V_k(0,j)\right\} + (1 - R_i)\cdot \operatorname{Min}\left\{C + \alpha \sum_j f_j V_k(0,j);\alpha \sum_j f_j V_k(i+1,j)\right\}.$$

For example, when $k = 0$ Equations (5a) and (5b) would yield

$$V_1(i,j) = R_i(K + C) \quad \text{for } j > 0$$

and,

$$V_1(i,0) = R_i \cdot C .$$

Intuitively, $V_k(i,j)$ is the cost if we follow policy π^* for k periods and incur a terminal cost of zero, given we start in state (i,j). Given $\alpha < 1$, it follows that

$$V_{k+1}(i,j) \geq V_k(i,j) \quad \text{for all } i, j, k \text{ and,}$$

$$\lim_{k\to\infty} V_k(i,j) = V(i,j) \quad \text{for all } i, j .$$

Equations (5a) and (5b) give an easy technique to evaluate V in terms of K, C, R, g and α. In addition, they will be helpful in establishing the monotonicity of V in i, the age of Unit 1.

2.2.3 Optimal Policy Structure

As a necessary prelude to establishing the structure of the optimal policy, we first show the monotonicity of $V(i,j)$ in the age of Unit 1.

Lemma 2.1:

If Unit 1 has increasing failure rate $(R_i \uparrow i)$, and K, $C > 0$, then for all k

(a) $V_k(i,j) \uparrow i$ for all j.

(b) $K + C + \alpha V_k(0,j) \geq \alpha V_k(i,j)$ for all i, j.

Proof:

For $k = 0$, $V_0(i,j) = 0$ for all i, j by definition. Therefore (a) and (b) are trivially true. The proof proceeds by induction on k. Suppose (a) and (b) are true for some k. First, we show $V_{k+1}(i,j) \uparrow i$ for all j.

Case (i): $j > 0$

Rewriting the recursive Equation (5a),

$$\begin{aligned}
V_{k+1}(i,j) &= R_i \{K+C+\alpha V_k(0,j-1\} + (1-R_i)\alpha V_k(i+1,j-1) \\
&= R_i \{K+C+\alpha V_k(0,j-1) - \alpha V_k(i+1,j-1)\} + \alpha V_k(i+1,j-1) \\
&\leq R_{i+1} \{K+C+\alpha V_k(0,j-1) - \alpha V_k(i+1,j-1)\} + \alpha V_k(i+1,j-1) \\
&= R_{i+1} \{K+C+\alpha V_k(0,j-1)\} + (1-R_{i+1}) \cdot \alpha V_k(i+1,j-1) \\
&\leq R_{i+1} \{K+C+\alpha V_k(0,j-1)\} + (1-R_{i+1}) \cdot \alpha V_k(i+2,j-1) \\
&= V_{k+1}(i+1,j) .
\end{aligned}$$

The first inequality is true since $R_{i+1} \geq R_i$ and the coefficient of R_i is nonnegative by the induction hypothesis (b). The second inequality follows from induction hypothesis (a), namely $V_k \uparrow i$.

Case (ii): $j = 0$

First note that

$$
\begin{aligned}
& V_k(i,j) \uparrow i \quad \text{for all } j \\
(6) \qquad & \Rightarrow \sum_j f_j V_k(i+1,j) \uparrow i \\
& \Rightarrow \text{Min}\left\{C + \alpha \sum_j f_j V_k(0,j); \alpha \sum_j f_j V_k(i+1,j)\right\} \uparrow i .
\end{aligned}
$$

Now,

$$
\begin{aligned}
V_{k+1}(i,0) &= R_i \left\{C + \alpha \sum_j f_j V_k(0,j)\right\} + (1 - R_i) \cdot \text{Min}\left\{C + \alpha \sum_j f_j V_k(0,j);\right. \\
&\qquad \left. \alpha \sum_j f_j V_k(i+1,j)\right\} \\
&\leq R_{i+1}\left\{C + \alpha \sum_j f_j V_k(0,j)\right\} + (1 - R_{i+1}) \cdot \\
&\qquad \text{Min}\left\{C + \alpha \sum_j f_j V_k(0,j); \alpha \sum_j f_j V_k(i+1,j)\right\} \\
&\leq R_{i+1}\left\{C + \alpha \sum_j f_j V_k(0,j)\right\} + (1 - R_{i+1}) \cdot \\
&\qquad \text{Min}\left\{C + \alpha \sum_j f_j V_k(0,j); \alpha \sum_j f_j V_k(i+2,j)\right\} \\
&= V_{k+1}(i+1,0) .
\end{aligned}
$$

The first inequality follows from $R_{i+1} \geq R_i$ and

$$
C + \alpha \sum_j f_j V_k(0,j) \geq \text{Min}\left\{C + \alpha \sum_j f_j V_k(0,j); \alpha \sum_j f_j V_k(i+1,j)\right\} .
$$

The second inequality follow from (6).

We have shown $V_{k+1}(i,j) \uparrow i$ for all j. It still remains to show that Part (b) is true for $(k+1)$. That is,

$$(7) \qquad K + C + \alpha V_{k+1}(0,j) \geq \alpha V_{k+1}(i,j) \quad \text{for all } i, j .$$

For $j > 0$,

$$\text{L.H.S.} = (K+C) + \alpha R_0 \{K+C+\alpha V_k(0,j-1)\} + \alpha^2(1-R_0) \cdot V_k(1,j-1)$$

$$= (K+C) + \alpha R_0(K+C) + \alpha^2 [R_0 V_k(0,j-1) + (1-R_0)V_k(1,j-1)]$$

$$\geq (K+C) + \alpha R_0(K+C) + \alpha^2 [R_0 V_k(0,j-1) + (1-R_0)V_k(0,j-1)]$$

(by the induction hypothesis $V_k \uparrow i$)

$$= (K+C) + \alpha R_0(K+C) + \alpha^2 V_k(0,j-1) = A$$

$$\text{R.H.S.} = \alpha [R_i \{K+C+\alpha V_k(0,j-1)\} + (1-R_i) \cdot \alpha V_k(i+1,j-1)]$$

$$\leq \alpha [R_i \{K+C+\alpha V_k(0,j-1)\} + (1-R_i) \cdot \{K+C+\alpha V_k(0,j-1)\}]$$

(by the induction hypothesis (b))

$$= \alpha(K+C) + \alpha^2 V_k(0,j-1)$$

$$< A , \text{ (by inspection)} .$$

When $j = 0$ Equation (7) follows similarly.■

When we are in state $(i,0)$ the optimal policy π^* chooses the action (replace Unit 1 or not) that minimizes the total expected cost from thereon. The next theorem determines its structure.

Theorem 2.2:

If Unit 1 is IFR, and K, $C > 0$, then there exists an $i^* \in (0,\infty)$ such that when in state $(i,0)$,

(a) replace Unit 1 if $i \geq i^*$

(b) do not replace Unit 1 if $i < i^*$

(c) $i^* = \operatorname{Min}\left\{i : \alpha \sum_j f_j V(i,j) > C + \alpha \sum_j f_j V(0,j)\right\}$.

Proof:

As $k \to \infty$, Lemma 2.1 implies $V(i,j) \uparrow i$ for all j. nerefore, $\alpha \sum_j f_j V(i,j) \uparrow i$.

Define $i^* = \operatorname{Min}\left\{i : \alpha \sum_j f_j V(i,j) > C + \alpha \sum_j f_j V(0,j)\right\}$.

Then, for all states $(i,0)$ where $i \geq i^*$,

$$\alpha \sum_j f_j V(i,j) > C + \alpha \sum_j f_j V(0,j)$$

and replacing Unit 1 minimizes the cost objective. The above structure of π^* follows. ■

To compute the optimal policy parameter i^*, let us define

$$\begin{aligned} i^*(k) &= \operatorname{Min}\left\{i : \alpha \sum_j f_j V_k(i,j) > C + \alpha \sum_j f_j V_k(0,j)\right\} \\ &= \operatorname{Min}\left\{i : \alpha \sum_j f_j (V_k(i,j) - V_k(0,j)) > C\right\}. \end{aligned}$$

It is unfortunate that $i^*(k)$ is not monotonic in k and cannot serve as a bound on i^*. We show this by a simple counterexample.

2.2.4 Extensions of the Model

One extension of the model is immediate. When Units 1 and 2 have continuous density functions, the above results are true for any discrete approximation of the failure rate function of Unit 1 and the density function of Unit 2, as well as for the limiting continuous functions.

To establish the optimal policy that minimizes the expected cost per unit time we indicate the approach suggested in (Ross [1970]). For the discounted cost model with discount factor α, we define

$$h_\alpha(i,j) = V(i,j) - V(0,0) .$$

The functional equations (4a) and (4b) can then be rewritten as

$$\text{(8a)} \quad \begin{aligned} (1-\alpha)V(0,0) + h_\alpha(i,j) = {} & R_i\,\{K + C + \alpha h_\alpha(0,j-1)\} + \\ & (1-R_i)\cdot \alpha h_\alpha(i+1,j-1) \quad \text{for } j > 0 \end{aligned}$$

and,

$$\text{(8b)} \quad \begin{aligned} (1-\alpha)V(0,0) + h_\alpha(i,0) = {} & R_i\left\{C + \alpha \sum_j f_j h_\alpha(0,j)\right\} + \\ & (1-R_i)\cdot \text{Min}\left\{C + \alpha \sum_j f_j h_\alpha(0,j);\ \alpha \sum_j f_j h_\alpha(i+1,j)\right\} . \end{aligned}$$

When K and C are finite we note that $V(i,j) - V(0,0) < \infty$ for all i, j and α. Under this condition, $h_{\alpha_n}(i,j)$ converges to a bounded function $h(i,j)$ and, $(1-\alpha_n)V(0,0)$ converges to a constant g for some sequence $\alpha_n \to 1$ (for proof see Ross [1970], Section 6.7). In the limit, Equations (8a) and (8b) become

$$g + h(i,j) = R_i\,\{K + C + h(0,j-1)\} + \tag{9a}$$
$$(1 - R_i) \cdot h(i+1, j-1) \quad \text{for } j > 0$$

and,

$$g + h(i,0) = R_i \left\{ C + \sum_j f_j h(0,j) \right\} + \tag{9b}$$
$$(1 - R_i) \cdot \mathrm{Min} \left\{ C + \sum_j f_j h(0,j);\ \sum_j f_j h(i+1,j) \right\} .$$

When Equations (9a) and (9b) uphold, the optimal policy π^* is stationary and the minimum average cost of maintenance is g .

To define π^* for the average cost objective, first note that

$$V(i,j) \uparrow i \text{ for all } j \Rightarrow h(i,j) \uparrow i \text{ for all } j$$
$$\Rightarrow \sum_j f_j h(i,j) \uparrow i .$$

In the likeness of Theorem 2.2 we define

$$i_\phi^* = \mathrm{Min} \left\{ i : \sum_j f_j h(i,j) > C + \sum_j f_j h(0,j) \right\} .$$

Then for all states $(i,0)$ where $i \geq i_\phi^*$,

$$\sum_j f_j h(i,j) > C + \sum_j f_j h(0,j)$$

and it is optimal to replace Unit 1.

2.2.5 Some Special Cases

When $f_0 = 1$

In formulating the discrete time discounted cost model we specifically preclude the possibility $f_0 = 1$. If we interpret f_i to be the density function for the repairman's interarrival times (see Example (1)), then $f_0 = 1$ implies that the repairman and the associated economic advantage in replacement is always present. This is the case in the periodic age replacement model. It suffices now to define the state of the system by the age of Unit 1, $i = 0,1,2, \ldots$. The functional equation governing the optimal discounted cost function $V(i)$ is

$$\begin{aligned} V(i) &= \text{Min } \{\text{Expected cost if we replace; Expected cost if we don't replace}\} \\ &= \text{Min } \{C + \alpha V(0); R_i(K + C + aV(0)) + (1 - R_i) \cdot \alpha V(i + 1)\} . \end{aligned}$$

When $R_i \uparrow i$, it is easy to show that $V(i)$ is monotonic in i and the optimal policy structure may be characterized by i^*: Replace upon failure or when the age of Unit 1 equals i^* where

$$i^* = \underset{i}{\text{Min}} \{C + \alpha V(0) < R_i(K + C + \alpha V(0)) + (1 - R_i)\alpha V(i + 1)\} .$$

As before, this result extends to the case of a continuous failure distribution. Using the principles of Renewal Theory, Fox [1966] shows the optimality of this policy for a continuous IFR distribution and a continuous discounting criterion.

Series System of Several Components

Consider a series system of several components where each component is immediately replaced when it fails. The failure of any component entails a cost for its replacement plus a fixed cost K for system interruption. Now suppose we want a replacement policy for some critical component in this system. We regard this component as Unit 1 and the rest of the system as Unit 2 and note that Example 2 describes this situation. Furthermore, the failure distribution for Unit 2 tends to the exponential as its complexity and the time of operation increase (see Barlow and Proschan [1965], Section 2.3). If past failure data is available, it is easy to estimate the parameter that defines this exponential distribution. Similarly, we may consider another critical components as Unit 1 and determine a replacement policy for it. This gives us an approximate technique to introduce an opportunistic policy for components where we may realize maximum benefit. Components whose replacement costs are small compared to K are good choices for an opportunistic policy.

The special case where opportunities to replace or not arrive as a Poisson process receives special attention in the next chapter. Using Renewal Theory, an explicit solution for the average cost optimal policy parameter i^* is derived.

Replacements with Different Life Distributions

Another situation of interest arises when the failure distribution for repaired components is different than for original components.

Let $m = 0$ if a new component is in service and $m = 1$ if a used component is in service. Let (i,j,m) describe the state of the system, and, $R_{i,m}$ and $V(i,j,m)$ be the associated failure rate and the cost objective respectively. The functional equations for this case are

$$V(i,j,m) = R_{i,m}\{K + C + \alpha V(0,j-1,1)\} + (1 - R_{i,m}) \cdot \alpha V(i+1,j-1,m)$$

$$\text{for } m = 0\ ,\ 1 \text{ and } j > 0$$

and,

$$V(i,0,m) = R_{i,m}\left\{C + \alpha \sum_j f_j V(0,j,1)\right\} + (1 - R_{i,m}) \text{ Min } \left\{C + \alpha \sum_j f_j V(0,j,1);\right.$$
$$\left.\alpha \sum_j f_j V(i+1,j,m)\right\} \text{ for } m = 0\ ,\ 1 \text{ and } j = 0\ .$$

If the new and the repaired units have increasing failure rates and, $R_{i,1} \geq R_{i,0}$ for all i, then there is a critical replacement age $i_0^*\left(i_1^*\right)$ for the new (repaired) unit and the optimal policy again has a simple structure as given in Theorem 2.2. The successive approximation technique can be used to evaluate the cost and approximate i_0^* and i_1^*.

2.3 Simultaneous Replacement of Two Components

2.3.1 Model Formulation

Consider a series system of two units, 1 and 2. The system fails when either unit fails and is repaired by replacing the failed unit immediately. If unit n fails we incur a cost of $(K + C_n)$ where C_n is replacement cost of component n and K is the

overhead cost associated with system failure. When the system is down, we have the option to replace the nonfailed unit at its marginal cost C_n. In replacing the nonfailed unit we sacrifice its remaining life but hope to forestall the next system failure.

Clearly, failure epochs of Unit 1 (Unit 2) are potential opportunities to replace Unit 2 (Unit 1). As in Section 2, we wish to determine an optimal stationary replacement policy that minimizes the total expected discounted cost of maintaining the entire system. Let V denote the minimum cost objective and π^* the "best" policy. Let α be the discount rate.

Suppose the system operates in discrete time $t = 0,1,2, \ldots$. Let (i,j) denote the state of the system where $i = 0,1,2, \ldots$ is the age of Unit 1 and $j = 0,1,2, \ldots$ is the age of Unit 2. Suppose that if Unit n fails (or is opportunistically replaced) at time t, it returns to state 0 at time $t + 1$. Let R_i (P_j) denote the discrete failure rate of Unit 1 (Unit 2) at age i (j). In this discrete time space we allow the simultaneous failure of both units. Given state (i,j), the minimum cost function $V(i,j)$ obeys the functional equation

$$\begin{aligned} V(i,j) = {} & R_iP_j\,\{K + C_1 + C_2 + \alpha V(0,0)\} + R_i\bar{P}_j\,\{K + C_1 + S(0,j+1)\} \\ & + \bar{R}_iP_j\,\{K + C_2 + T(i+1,0)\} + \bar{R}_i\bar{P}_j\,\{\alpha V(i+1,j+1)\} \end{aligned} \tag{10}$$

where $\bar{R}_i \equiv 1 - R_i$ and $\bar{P}_j \equiv 1 - P_j$, and where

$$S(0,j+1) = \mathrm{Min}\,\{C_2 + \alpha V(0,0);\alpha V(0,j+1)\} \tag{11}$$

$$T(i+1,0) = \mathrm{Min}\,\{C_1 + \alpha V(0,0);\alpha V(i+1,0)\} \tag{12}$$

If Unit 1 fails and Unit 2 does not, we would take the opportunity and replace 2 only if $S(0,j+1) = C_2 + \alpha V(0,0)$. If 2 fails and 1 does not, we would take the opportunity and replace 1 only if $T(i+1,0) = C_1 + \alpha V(0,0)$.

Given the costs K , C_1 , and C_2 , and failure rate functions R_i and P_j , we can calculate $V(i,j)$ by using the successive approximation technique again:

$$V_{k+1}(i,j) = R_i P_j \{K + C_1 + C_2 + \alpha V_k(0,0)\} + R_i \bar{P}_j \{K + C_1 + S_k(0,j+1)\}$$

$$\text{(13)} \qquad + \bar{R}_i P_j \{K + C_2 + T_k(i+1,0)\} + \bar{R}_i \bar{P}_j \{\alpha V_k(i+1,j+1)\}$$

where

$$\text{(14)} \qquad S_k(0,j+1) = \text{Min}\,\{C_2 + \alpha V_k(0,0);\alpha V_k(0,j+1)\}$$

$$\text{(15)} \qquad T_k(i+1,0) = \text{Min}\,\{C_1 + \alpha V_k(0,0);\alpha V_k(i+1,0)\} .$$

Define $V_0(i,j) = 0$ for all (i,j) . Then $S_0(0,j+1) = 0$ for all j , and $T_0(i+1,0) = 0$ for all i .

Intuitively, $V_k(i,j)$ is the expected discounted cost if we follow policy π^* for k periods and end with zero terminal cost. For K , C_1 , and $C_2 < \infty$, and $\alpha < 1$,

$$\lim_{k\to\infty} V_k(i,j) = V(i,j)$$

$$\lim_{k\to\infty} S_k(0,j+1) = S(0,j+1)$$

$$\lim_{k\to\infty} T_k(i+1,0) = T(i+1,0) .$$

Also note that $V_k(i,j) \uparrow k$. Besides its computational value, this technique helps us to establish the monotonicity of V in i and j when Unit 1 and Unit 2 are IFR. This ultimately leads to a simple structure for the best replacement policy π^* .

2.3.2 Optimal Policy Structure

Lemma 3.1:

If Unit 1 and Unit 2 are both IFR ($R_i \uparrow i$ and $P_j \uparrow j$) , and K , C_1 , and $C_2 > 0$, then for all k

(a) $V_k(i,j) \uparrow i$ for all j

(b) $V_k(i,j) \uparrow j$ for all i .

Proof:

For $k = 0$, (a) and (b) are trivially true since $V_k(i,j) = 0$ for all i , j . Also observe that for $k = 0$,

(a') $K + C_2 + T_k(i + 1,0) \geq \alpha V_k(i + 1,j + 1)$, and

(b') $K + C_1 + S_k(0,j + 1) \geq \alpha V_k(i + 1,j + 1)$.

Intuitively, (a') and (b') imply that no failure entails no more cost than failure of either unit for a k period problem.

The proof proceeds by induction on k . Suppose (a), (b), (a'), and (b') are true for some k . First, we show that $V_{k+1}(i,j) \uparrow j$ for all i . Rewriting the iterative equation (13),

$$V_{k+1}(i,j) = R_i[P_j\{K + C_1 + C_2 + \alpha V_k(0,0)\} + \bar{P}_j\{K + C_1 + S_k(0,j + 1)\}]$$

$$+ \bar{R}_i[P_j\{K + C_2 + T_k(i + 1,0)\} + \bar{P}_j\{\alpha V_k(i + 1,j + 1)\}]$$

$$= R_i A(j) + \bar{R}_i B(j) \ ,$$

where $A(j)$ and $B(j)$ are functions of j , given some value of i . We proceed to show that both $A(j)$ and $B(j) \uparrow j$ for all i .

$$A(j) = P_j \ \{K + C_1 + C_2 + \alpha V_k(0,0)\}$$

$$+ \bar{P}_j \cdot \mathrm{Min}\ \{K + C_1 + C_2 + \alpha V_k(0,0);K + C_1 + \alpha V_k(0,j + 1)\} \ .$$

Since $P_j \uparrow j$, and

$$K + C_1 + C_2 + \alpha V_k(0,0)$$

$$\geq \mathrm{Min}\ \{K + C_1 + C_2 + \alpha V_k(0,0);K + C_1 + \alpha V_k(0,j + 1)\} \ ,$$

therefore

$$\begin{aligned} A(j) &\leq P_{j+1}\ \{K + C_1 + C_2 + \alpha V_k(0,0)\} \\ &\quad + \bar{P}_{j+1} \cdot \mathrm{Min}\ \{K + C_1 + C_2 + \alpha V_k(0,0);K + C_1 + \alpha V_k(0,j + 1)\} \\ &\leq P_{j+1}\ \{K + C_1 + C_2 + \alpha V_k(0,0)\} \\ &\quad + \bar{P}_{j+1} \cdot \mathrm{Min}\ \{K + C_1 + C_2 + \alpha V_k(0,0);K + C_1 + \alpha V_k(0,j + 2)\} \\ &\quad \text{(since } V_k \uparrow j \text{ for all } i \text{ by the induction hypothesis (b))} \\ &= A(j + 1) \ . \end{aligned}$$

Now,

$$B(j) = P_j \{K + C_2 + T_k(i+1,0)\} + \bar{P}_j \{\alpha V_k(i+1,j+1)\} .$$

Since $P_j \uparrow j$, and by the induction hypothesis (a')

$$K + C_2 + T_k(i+1,0) \geq \alpha V_k(i+1,j+1) ,$$

therefore

$$\begin{aligned} B(j) &\leq P_{j+1} \{K + C_2 + T_k(i+1,0)\} + \bar{P}_{j+1} \{\alpha V_k(i+1,j+1)\} \\ &\leq P_{j+1} \{K + C_2 + T_k(i+1,0)\} + \bar{P}_{j+1} \{\alpha V_k(i+1,j+2)\} \\ &\quad (\text{since } V_k \uparrow j \text{ for all } i) \\ &= B(j+1) . \end{aligned}$$

Hence, $V_{k+1}(i,j) = [R_i A(j) + \bar{R}_i B(j)] \uparrow j$ for all i . Similarly, we can show that $V_{k+1}(i,j) \uparrow i$ for all j .

It still remains to show that assumptions (a') and (b') extend to $k+1$. That is,

(17) (a') $K + C_2 + T_{k+1}(i+1,0) \geq \alpha V_{k+1}(i+1,j+1)$ for all i , j .

(18) (b') $K + C_1 + S_{k+1}(0,j+1) \geq \alpha V_{k+1}(i+1,j+1)$ for all i , j .

Equation (17) restated is

$$K + C_2 + \text{Min}\,\{C_1 + \alpha V_{k+1}(0,0); \alpha V_{k+1}(i+1,0)\} \geq \alpha V_{k+1}(i+1,j+1) ,$$

which is equivalent to

(19a) $K + C_1 + C_2 + \alpha V_{k+1}(0,0) \geq \alpha V_{k+1}(i+1,j+1)$, and

(19b) $K + C_2 + \alpha V_{k+1}(i+1,0) \geq \alpha V_{k+1}(i+1,j+1)$.

To see (19a) note that

$$\alpha V_{k+1}(i + 1, j + 1)$$

$$= \alpha[R_{i+1}P_{j+1}\ \{K + C_1 + C_2 + \alpha V_k(0,0)\} + R_{i+1}\bar{P}_{j+1}\ \{K + C_1 + S_k(0, j + 2)\}$$

$$+ \bar{R}_{i+1}P_{j+1}\ \{K + C_2 + T_k(i + 2,0)\} + \bar{R}_{i+1}\bar{P}_{j+1}\ \{\alpha V_k(i + 2, j + 2)\}]$$

$$\leq \alpha[K + C_1 + C_2 + \alpha V_k(0,0)]$$

$$\leq K + C_1 + C_2 + \alpha V_k(0,0)\ , \text{ since } \alpha < 1$$

$$\leq K + C_1 + C_2 + \alpha V_{k+1}(0,0)\ , \text{ since } V_k \uparrow k\ .$$

For Equation (19b), the left hand side is

$$(K + C_2) + \alpha V_{k+1}(i + 1,0)$$

$$= (K + C_2) + \alpha\ \{R_{i+1}P_0\alpha V_k(0,0) + R_{i+1}\bar{P}_0 S_k(0,1) + \bar{R}_{i+1}P_0 T_k(i + 2,0)$$

$$+ \bar{R}_{i+1}\bar{P}_0\alpha V_k(i + 2,1) + R_{i+1}C_1 + P_0C_2 + (R_{i+1} + P_0 - R_{i+1}P_0)K\}$$

$$\geq (K + C_2) + \alpha\ \{R_{i+1}\alpha V_k(0,0) + \bar{R}_{i+1}T_k(i + 2,0)$$

$$+ R_{i+1}C_1 + P_0C_2 + (R_{i+1} + P_0 - R_{i+1}P_0)K\}$$

$$= \beta\ .$$

The inequality follows from $\alpha V_k(0,0) \leq S_k(0,1)$ and $T_k(i + 2,0) \leq \alpha V_k(i + 2,1)$. The right hand side of (19b) is

$$\alpha V_{k+1}(i + 1, j + 1) \leq \alpha\ \{R_{i+1}(K + C_1 + C_2 + \alpha V_k(0,0)$$

$$+ \bar{R}_{i+1}(K + C_2 + T_k(i + 2,0))\}$$

$$\leq \beta \quad \text{(by inspection)}\ .$$

The first inequality follows from $K + C_1 + C_2 + V_k(0,0) \geq K + C_1 + S_k(0,j+2)$ and by induction hypothesis (a'). Therefore, the L.H.S. $\geq \beta \geq$ R.H.S. of Equation (19b). The proof for Equation (18) is analogous. ∎

If we let $k \to \infty$ in Lemma 3.1, we get $V(i,j) \uparrow i$ for all j, and $V(i,j) \uparrow j$ for all i. This leads us to the simple replacement policy given in the next theorem.

Theorem 3.2:

If Units 1 and 2 are IFR, and K, C_1, and $C_2 > 0$, then there exists i^* and j^* such that

(a) If Unit 1 fails and $j \geq j^*$, replace both units. Otherwise replace only the failed Unit 1.

(b) If Unit 2 fails and $i \geq i^*$, replace both units. Otherwise, replace only the failed Unit 2.

(c) $j^* = \text{Min}\,\{j : V(0,j) > C_2 + \alpha V(0,0)\}$.

(d) $i^* = \text{Min}\,\{i : V(i,0) > C_1 + \alpha V(0,0)\}$.

Proof:

Define j^* and i^* as above. From Lemma 3.1, $V(0,j) \uparrow j$ and $V(i,0) \uparrow i$. Therefore for all $j \geq j^*$, $V(0,j) > C_2 + \alpha V(0,0)$ and Unit 2 should be opportunistically replaced. Similarly, for all $i \geq i^*$, $V(i,0) > C_1 + \alpha V(0,0)$ which implies Unit 1 should be opportunistically replaced. The above structure of the optimal policy π^* follows. ∎

In practice the critical replacement ages i^* and j^* can be approximated by using the successive approximation technique for a large number of iterations.

2.3.3 Extensions of the Model

When Units 1 and 2 have continuous IFR distributions, the preceding results are again true for any discrete approximations of the failure rate functions as well as for the limiting continuous functions. When the expected average cost of maintenance is to be minimized, a sufficient condition for a stationary optimal policy to exist is

$$V(i,j) - V(0,0) < N < \infty .$$

When costs K, C_1, and C_2 are finite, this condition is obviously upheld. Using the technique suggested in Section 2, we can again show that the optimal policy inherits the structure from the expected discounted cost case when both units are IFR.

The simplicity of the optimal policy structure for the two models considered does not extend to more complicated systems with interactive replacement activities. For example, let (i,j,l) be the age vector for a system of three units in series. If Unit 1 fails, the decision to replace Unit 2 (Unit 3) would also depend on the age of Unit 3 (Unit 2) and the associated cost advantages thereof.

For convenience, we consider a series system of Unit 1 and Unit 2. Unit 2 has an exponential life distribution and its failures are potential opportunities to replace Unit 1. If Unit 2 fails and the age of Unit 1 exceeds 'a', we replace both units. In any case the failed unit is immediately replaced. Let $(K + C)$ be the cost of a failure replacement and C be the cost of an opportunistic replacement for Unit 1. Under such a policy, Section 2 examines the time between successive failures, successive replacements and other quantities of interest. In Section 3, an explicit solution for the average cost optimal policy is derived. Section 4 examines certain classes of distributions for Unit 1 which have received wide attention in other replacement models.

3.2 Failure and Replacement Intervals

Let $F(x)$ be the failure distribution function of Unit 1 and, $G(x) = 1 - e^{-x/\mu}$ be the failure distribution of Unit 2. We first observe that the replacement of Unit 1, be it opportunistic or due to its failure, is a renewal for the entire system. This is so because when Unit 2 is working, it is as good as new. If Z denotes the random time between successive replacements of Unit 1 and H is the corresponding distribution function, then

$$\text{(1)} \qquad \text{Prob}\ \{Z > x\} = \bar{H}(x) = \begin{cases} \bar{F}(x) & \text{for } x < a \\ \bar{F}(x) \cdot \bar{G}(x - a) & \text{for } x \geq a \end{cases}$$

where $\bar{H} = 1 - H$ and, this notation is followed for all other distribution functions as well. The mean time between successive replacements can be expressed as

$$E[Z] = \int_0^\infty \bar{H}(x)dx$$

(2)

$$= \int_0^a \bar{F}(x)dx + \int_a^\infty \bar{F}(x)\bar{G}(x - a)dx \ .$$

A replacement is opportunistic if Unit 1 is working and has survived for at least time 'a' when Unit 2 fails. If P_0 is the probability that any replacement is opportunistic, then

(3) $$P_0 = \int_0^\infty \bar{F}(x + a)dG(x) \ .$$

To find the expected time between opportunistic replacements we invoke Wald's equation (see Ross [1970]). Let $Z_1, Z_2, Z_3, \ldots$ be a sequence of intervals between successive replacements, and suppose that the N^{th} replacement is the first opportunistic replacement. The Z_i's are independent with a common distribution given by Equation (1). Hence the expected time between opportunistic replacement can be expressed as

$$E\left[\sum_1^N Z_i\right] = E[Z] \cdot E[N] = E[Z]/P_0$$

provided the random variables have finite expectations. Similarly, the expected time between failure replacements of Unit 1 is $E[Z]/(1 - P_0)$.

The distribution of the time between successive opportunistic replacements is a little more involved. Let Z_r denote this

random interval and G be its distribution function. Since we cannot have an opportunistic replacement before 'a', therefore $Z_r \geq a$. We condition on the first failure of Unit 1, denoted by X_1, which is also a renewal for the system and, observe

$$\text{Prob}\{Z_r > t \mid X_1 = x\} = \begin{cases} \bar{G}(t - x) & \text{for } x < a \leq t \\ \bar{G}(t - x) \cdot \bar{G}(x - a) & a \leq x < t \\ \bar{G}(t - a) & a \leq t \leq x \,. \end{cases}$$

Unconditioning, we obtain

$$(4) \quad \bar{G}(t) = \begin{cases} 1 & \text{for } t < a \\ \bar{G}(t - a)\bar{F}(t) + \int_0^t \bar{G}(t - x)\bar{G}(x - a)dF(x) & \text{for } t \geq a \end{cases}$$

where $\bar{G}(x - a) = 1$ when $x \leq a$.

Now let Z_f denote the random interval between failure replacements of Unit 1 and F be its distribution. When $t < a$, the only kind of replacement possible is a failure replacement. Therefore, $\bar{F}(t) = \bar{F}(t)$ for $t < a$. For $t \geq a$, we condition on the first failure of the exponential unit after time 'a' and, derive a renewal type equation for F as we did for G. We obtain

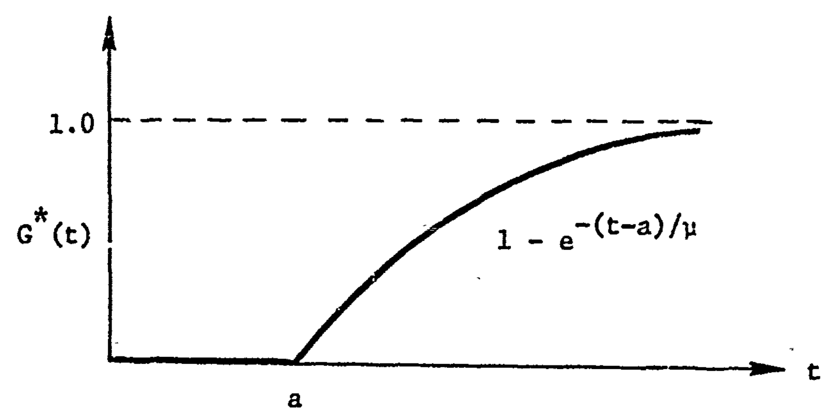
$$(5) \quad \bar{F}(t) = \begin{cases} \bar{F}(t) & \text{for } t < a \\ \bar{F}(t)\bar{G}(t - a) + \int_0^{t-a} \bar{F}(x + a)\bar{F}(t - x - a)dG(x) & \text{for } t \geq a \,. \end{cases}$$

Equations (4) and (5) define G and F implicitly and, in principle, can be solved by Laplace Transforms. For the various replacement models in the literature, these and other operating

characteristics have been studied by Barlow and Proschan in [1962], by Flehinger in [1962], and by McCall in [1963].

An analogy with a random age replacement policy is due. Under such a policy, we replace the unit at time T or at failure, whichever comes first. We assume that T has a known distribution G^*. Let us define G^* as

$$G^*(t) = \begin{cases} 0 & \text{for } t < a \\ G(t-a) & \text{for } t \geq a \end{cases}$$



where G is the exponential distribution corresponding to Unit 2 in the opportunistic model. The renewal processes associated with this age replacement and the opportunistic model now have the same distributions.

3.3 Optimal Average Cost Policy

The structure of the opportunistic policy, as characterized by parameter 'a', was shown to be optimal when Unit 1 is IFR in Chapter 2, Section 2. For the special case of an IFR Unit 1 in series with an exponential Unit 2 we can derive an explicit solution for the optimal value of 'a'.

From Renewal Reward Theory, the long-run expected cost per unit time, denoted by $\phi(a)$, may be expressed as

$$\phi(a) = \frac{\text{Expected Cost of a Renewal Cycle}}{\text{Expected Length of a Renewal Cycle}} \tag{6}$$

$$= \frac{P_0 C + (1 - P_0) \cdot (K + C)}{E[Z]}$$

where P_0 is the probability that a replacement is opportunistic. A necessary condition for the optimal value of 'a' is obtained by equating the derivative of $\phi(a)$ to zero. Thereby, we obtain

$$\frac{d}{da}\left(\frac{C + K(1 - P_0)}{E[Z]}\right) = 0$$

or,

$$KE[Z] \cdot \frac{dP_0}{da} + C \cdot \frac{dE[Z]}{da} + K(1 - P_0) \cdot \frac{dE[Z]}{da} = 0 \; . \tag{7}$$

From Equations (2) and (3), it is easy to verify that $(dE[Z]/da) = P_0$. Using this fact and dividing Equation (7) by KP_0, we get

$$\frac{E[Z]}{P_0} \cdot \frac{dP_0}{da} + (1 - P_0) = \frac{C}{K} .$$

Substituting the expressions derived in Section 2 and simplifying, we obtain

$$(8) \qquad \int_0^a \bar{F}(x)dx \cdot \frac{\int_0^\infty f(x + a)dG(x)}{\int_0^\infty \bar{F}(x + a)dG(x)} - F(a) = \frac{C}{K}$$

where f is the failure density for Unit 1.

It is illustrative to note the similarity of this condition with the one derived for a fixed age replacement policy:

$$(9) \qquad \int_0^a \bar{F}(x)dx \cdot \frac{f(a)}{\bar{F}(a)} - F(a) = \frac{C}{K}$$

where we interpret K as a savings in a planned replacement over a failure replacement.

Let $r(x) = f(x)/\bar{F}(x)$ represent the failure rate function of Unit 1. If we assume that $r(x)$ is continuous and increasing, then the left hand side of Equation (8) is increasing. Therefore, the optimal policy as defined by Equation (8) does exist. If the optimal $a^* = \infty$, we infer that a policy of no opportunistic replacements is best. Let

(10) $$\beta(a) = \frac{\int_0^\infty f(x+a)dG(x)}{\int_0^\infty \bar{F}(x+a)dG(x)} = \frac{\int_a^\infty r(y)\bar{F}(y)dG(y)}{\int_a^\infty \bar{F}(y)dG(y)}$$

where $y = x + a$. It remains to verify that $r(x) \uparrow x$ implies

$$\frac{d}{da}\left(\int_0^a \bar{F}(x)dx \cdot \beta(a) - F(a)\right) \geq 0 \ .$$

Or,

(11) $$(\bar{F}(a) \cdot \beta(a) - f(a)) + \int_0^a \bar{F}(x)dx \cdot \left(\frac{d\beta(a)}{da}\right) \geq 0 \ .$$

From Equation (10) we see that $\beta(a) \geq r(a)$. Therefore, the first term in Equation (11) is positive. For the second term in Equation (11) we need only show that

$$\frac{d\beta(a)}{da} \geq 0 \ .$$

Or,

$$\frac{d}{da}\left(\frac{\int_a^\infty r(y)\bar{F}(y)dG(y)}{\int_a^\infty \bar{F}(y)dG(y)}\right) \geq 0 \ .$$

Or,

$$\bar{F}(a)g(a) \cdot \int_{a}^{\infty} r(y)\bar{F}(y)dG(y) - r(a)\bar{F}(a)g(a) \int_{a}^{\infty} \bar{F}(y)dG(y) \geq 0 \ .$$

Or,

$$\int_{a}^{\infty} r(y)\bar{F}(y)dG(y) \geq r(a) \int_{a}^{\infty} \bar{F}(y)dG(y)$$

which is obvious given $r(y) \uparrow y$. When the failure rate is strictly increasing, the optimal parameter a^* is finite.

3.4 Distributions in Replacement

Any maintenance policy is necessarily aimed against the incidence of actual failures of the unit under care. Under an opportunistic policy as defined in this chapter, we would like to establish a class of failure distributions which reduces the number of failures of Unit 1 in a given time t , either stochastically or in expected value. Fortunately, the classes of distributions relevant to this replacement model are already well known in Reliability Theory. Marshall and Proschan have studied these classes and established their importance in the Age and Block Replacement models in [1970]. First, we recall that a distribution F , with density f , has increasing failure rate, IFR, if $r(t) = f(t)/\bar{F}(t)$ is increasing in t . The other classes for F are defined below:

A distribution F is New Better than Used, NBU (New Worse than Used, NWU) if

$$(12) \qquad \bar{F}(x + y) \underset{(\geq)}{\leq} \bar{F}(x) \cdot \bar{F}(y) \quad \text{for all } x , y \geq 0 .$$

A distribution F , with mean θ , is New Better than Used in Expectation, NWUE, (New Worse than Used in Expectation, NWUE) if

$$(13) \qquad \int_0^\infty \bar{F}(x + y)dx \underset{(\geq)}{\leq} \bar{F}(y) \cdot \theta \quad \text{for all } y \geq 0 .$$

Each of the above classes can be interpreted as a notion of aging and one can establish the following hierarchy amongst them, (see Barlow and Proschan [1975]):

$$\text{IFR} \Rightarrow \text{NBU} \Rightarrow \text{NBUE}$$

and,

$$\text{DFR} \Rightarrow \text{NWU} \Rightarrow \text{NWUE} .$$

Let $\{X_i\}$ denote the sequence of intervals between successive failures of Unit 1 when no replacement policy is in effect, F denote their distribution and, N(t) denote the number of failures in (0,t) . Let $\{\mathcal{X}_i\}$ denote the intervals between in-service failures of Unit 1 under an opportunistic policy. The $\{\mathcal{X}_i\}$ have a common distribution $\mathcal{F}$ defined by Equation (5). Correspondingly, $\mathcal{N}(t)$ shall denote the number of failures in (0,t) . In general, $\mathcal{X}_i$, $\mathcal{F}$, and $\mathcal{N}$ depend on the policy parameter 'a' and, wherever necessary, we shall specify as $\mathcal{X}_i(a)$, $\mathcal{F}(\cdot,a)$ and $\mathcal{N}(t,a)$.

For the sake of results that follow immediately, we let $\{X_{i,k}\}$ denote the intervals between in-service failures of Unit 1 when at most k replacements are permitted between successive failures. Under this constrained opportunistic policy we may define F_k and $N_k(t)$ correspondingly. In our propositions that relate F with the in-service failure process, we shall use induction on k and bear in mind that $F_0 = F$, and $F_k \to F$ as $k \to \infty$. Equivalent to Equation (5), we define the in-service failure distribution under the constrained policy as

$$(14) \quad \bar{F}_k(t) = \begin{cases} \bar{F}(t) & \text{for } t < a \\ \bar{F}(t)\bar{G}(t-a) + \int_0^{t-a} \bar{F}(x+a)\bar{F}_{k-1}(t-x-a)\,dG(x) & \text{for } t \geq a \end{cases}$$

Our first result shows that the NBU is the largest class which stochastically increases the interval between in-service failures when we institute an opportunistic policy.

<u>Theorem 3.1:</u>

The failure distribution for Unit 1, F, is NBU (NWU) $\Leftrightarrow$ $x_{i,k}$ stochastically increases (decreases) in k for all exponential G associated with Unit 2, and policy parameter $a \geq 0$.

<u>Proof:</u>

First we show that F is NBU implies $\bar{F}_{k+1}(t) \geq \bar{F}_k(t)$ for $k = 0,1,2, \ldots$. Since the $x_{i,k}$ are independent and identically distributed, it suffices to prove the theorem for $i = 1$. Let $k = 0$. Then by Equation (14), we have

$$\bar{F}_1(t) = \bar{F}(t) \quad \text{for} \quad t < a$$

and,

$$\bar{F}_1(t) = \bar{F}(t) \cdot \bar{G}(t - a) + \int_0^{t-a} \bar{F}(x + a) \cdot \bar{F}_0(t - x - a)dG(x) \quad \text{for} \quad t \geq a .$$

Let us write $\bar{F}(t) \cdot \bar{G}(t - a)$ as $\bar{F}(t) - \int_0^{t-a} dG(x)$. Then,

$$\bar{F}_1(t) = \bar{F}(t) + \int_0^{t-a} [\bar{F}(x + a) \cdot \bar{F}(t - x - a) - \bar{F}(t)]dG(x)$$

$$\geq \bar{F}(t) \text{ , since the integrand is nonnegative}$$
$$\text{by the definition of NBU .}$$

When $k = 1$, we have

$$\bar{F}_2(t) = \bar{F}(t) \quad \text{for} \quad t < a$$

and,

$$\bar{F}_2(t) = \bar{F}(t) + \int_0^{t-a} [\bar{F}(x + a) \cdot \bar{F}_1(t \cdot x - a) - \bar{F}(t)]dG(x) \quad \text{for} \quad t \geq a$$

$$\geq \bar{F}(t) + \int_0^{t-a} [\bar{F}(x + a) \cdot \bar{F}(t - x - a) - \bar{F}(t)]dG(x) \quad \text{since}$$

$$\bar{F}_1(t - x - a) \geq \bar{F}(t - x - a) \quad \text{by the result proven above}$$

$$= \bar{F}_1(t) .$$

The proof proceeds identically by induction on k.

To show the converse, we assume $\bar{F}_{k+1}(t) - \bar{F}_k(t) \geq 0$ for all t. It suffices to consider the case when $k = 0$ and $t \geq a$. Then,

$$\bar{F}_1(t) - \bar{F}(t) = \int_0^{t-a} [\bar{F}(x + a) \cdot \bar{F}(t - x - a) - \bar{F}(t)]dG(x) \geq 0$$

for all $G(x) = 1 - e^{-x/\mu}$ and $a > 0$. If we let $\mu \to 0$ the exponential approaches the degenerate distribution and the above inequality becomes

$$\bar{F}(a) \cdot \bar{F}(t - a) - \bar{F}(t) \geq 0 \quad \text{for all } a > 0 , t \geq a .$$

Therefore F is NBU. When F is NWU all the inequalities in the proof are reversed. ∎

Corollary 3.2:

F is NBU (NWU) $\Leftrightarrow N_k(t)$ stochastically decreases (decreases) in k, for all exponential G and $a > 0$.

Proof:

Since the $\{X_{i,k}\}$ and $\{X_{i,k+1}\}$ are sequences of independant random variables, it follow from Theorem 3.1 that

$$\begin{aligned} \text{Prob } \{N_{k+1}(t) \geq n\} &= \text{Prob } \{X_{1,k+1} + \cdots + X_{n,k+1} \leq t\} \\ &\leq \text{Prob } \{X_{1,k} + \cdots + X_{n,k} \leq t\} \\ &= \text{Prob } \{N_k(t) \geq n\} . \end{aligned}$$

For the converse $N_{k+1}(t) \underset{st}{\leq} N_k(t)$. Therefore, for $k = 0$ we have

$$\text{Prob}\,\{N_1(t) = 0\} = \bar{F}_1(t) \geq \text{Prob}\,\{N(t) = 0\} = \bar{F}(t)\ .$$

Hence F is NBU by Theorem 3.1 again. When F is NWU all the inequalities are reversed. ■

Our primary interest was in prescribing minimal conditions on the distribution F which stochastically reduce the number of in-service failures in $(0,t)$ when an opportunistic policy is introduced. As a direct result of the above theorem, we can say that when F is NBU, $X_i \underset{st}{\geq} X_i$ and $N(0,t) \underset{st}{\leq} N(0,t)$ for any policy parameter 'a' and any Poisson arrival process for replacement opportunities.

As noted earlier, the NBUE is a weaker notion of aging than NBU. Correspondingly, we can establish that the NBUE class is the largest which can realize a reduction in the expected number of in-service failures when an opportunistic policy is introduced. The next theorem shall prove this.

<u>Theorem 3.3</u>:

The failure distribution for Unit 1, F, is NBUE (NWUE) $\Leftrightarrow$ $E[X_{i,k}]$ increases (decreases) in k for all exponential G associated with Unit 2, and policy parameter $a > 0$.

<u>Proof</u>:

First we show that F is NBUE implies $E[X_{1,k+1}] \geq E[X_{1,k}]$ for $k = 0,1,2,\ldots$. Since the $\{X_{i,k}\}$ are i.i.d it suffices

Since the integrand in expression (17) is nonnegative by the definition of NBUE, therefore

$$E[X_{1,1}] \geq E[X] .$$

Now let $k = 1$. Corresponding to expression (16), we get

$$E[X_{1,2}] = E[X] + \int_0^\infty \int_{x+a}^\infty [\bar{F}(x + a) \cdot \bar{F}_1(t - x - a) - \bar{F}(t)]dt \cdot dG(x)$$

$$= E[X] + \int_0^\infty \left[\bar{F}(x + a) \cdot E[X_{1,1}] - \int_{x+a}^\infty \bar{F}(t)dt\right]dG(x)$$

$$\left(\text{where we note that } \int_{x+a}^\infty \bar{F}_1(t - x - a)dt = E[X_{1,1}]\right)$$

$$\geq E[X] + \int_0^\infty \left[\bar{F}(x + a) \cdot E[X] - \int_{x+a}^\infty \bar{F}(t)dt\right]dG(x)$$

(since $E[X_{1,1}] \geq E[X]$ by the result proven above)

$$= E[X_{1,1}] .$$

Similarly, the proof proceeds on induction to show that $E[X_{1,k+1}] \geq E[X_{1,k}]$ for all k .

To show the converse, we assume $E[X_{1,1}] - E[X] \geq 0$ for all $a > 0$, and $G(x) = 1 - e^{-x/\mu}$. Substituting for these expected values, we get

$$\int_0^\infty \left[\bar{F}(x + a) \cdot E[X] - \int_{x+a}^\infty \bar{F}(t)dt\right]dG(x) \geq 0 .$$

If we let $\mu \to 0$, the exponential approaches the degenerate distribution and the above inequality becomes

$$\bar{F}(a) \cdot E[X] - \int_a^\infty \bar{F}(t)dt \geq 0 \quad \text{for all} \quad a > 0 .$$

Hence, F is NBUE. The results for NWUE follow by reversing all inequalities. ∎

Again, we may note the implication of the above theorem that is particularly interesting. If Unit 1 has distribution F that is NBUE, then the expected time between in-service failures increases when we introduce an opportunistic policy with any parameter 'a' and any Poisson arrival process for replacement opportunities.

We now assume that F conforms to the strongest notion of aging defined before, namely the IFR class. Under this assumption, the optimality of the policy structure was proven in Chapter 2. We shall now investigate the changes in the time between successive in-service failures with respect to changes in the policy parameter 'a'. First, we give an alternative definition of IFR:

$$F \text{ is IFR} \Leftrightarrow \frac{\bar{F}(t+\Delta)}{\bar{F}(t)} \text{ decreases in } t$$

for each $\Delta \geq 0$. This property of the survival function is called polya frequency of order two, PF_2 (see Barlow and Proschan [1975]).

We begin by establishing an intermediate result in the next lemma.

Lemma 3.4:

If Unit 1 has distribution F that is IFR, then

$$\bar{F}_k(t) \leq \bar{F}(a)\bar{F}_k(t-a)\ ,\ t \geq a\ ,\ k = 0,1,2,\ \ldots$$

for any policy parameter $a > 0$ and exponential G associated with Unit 2.

Proof:

For $k = 0$, $\bar{F}_0(t) \equiv \bar{F}(t)$. Since IFR $\Rightarrow$ NBU, the result is true by the definition of NBU.

Now assume the lemma is true for some $k > 0$. It remains to extend the result to $k + 1$. First note that

$$\bar{F}_{k+1}(t) = \bar{F}_k(t) \qquad \text{for} \quad t < (k+1)a$$

since it is impossible to use the $k + 1$ replacement opportunities allowed in time $t < (k+1)a$. Therefore, consider $t \geq (k+1)a$. Then, by Equation (14),

$$\bar{F}_{k+1}(t) = \bar{F}(t) \cdot \bar{G}(t-a) + \int_0^{t-a} \bar{F}(x+a) \cdot \bar{F}_k(t-x-a)dG(x)$$

(16)
$$= \bar{F}(t) \cdot \bar{G}(t-a) + \int_0^{t-2a} \bar{F}(x+a) \cdot \bar{F}_k(t-x-a)dG(x) + \int_{t-2a}^{t-a} \bar{F}(x+a) \cdot \bar{F}_k(t-x-a)dG(x)\ .$$

Now for $x \in (t-2a, t-a)$,

$$\bar{F}(x + a) \cdot \bar{F}_k(t - x - a) = \bar{F}(x + a) \cdot \bar{F}(t - x - a)$$
$$\leq \bar{F}(a) \cdot \bar{F}(t - a) \quad \text{since } \bar{F} \text{ is } PF_2 .$$

Substituting this inequality in the right hand side of Equation (18), we get

$$\bar{F}_{k+1}(t) \leq \bar{F}(t)\bar{G}(t - a) + \int_0^{t-2a} \bar{F}(x + a)\bar{F}_k(t - x - a)dG(x)$$
$$+ \bar{F}(a)\bar{F}(t - a)\{\bar{G}(t - 2a) - \bar{G}(t - a)\}$$
$$\leq \bar{F}(a)\bar{F}(t - a)\bar{G}(t - 2a) + \int_0^{t-2a} \bar{F}(x + a)\bar{F}_k(t - x - 2a)\bar{F}(a)dG(x)$$

(where we use the induction hypothesis $\bar{F}_k(t - x - a) \leq \bar{F}(a) \cdot \bar{F}_k(t - x - 2a)$)

$$= \bar{F}(a) \cdot \bar{F}_{k+1}(t - a) . \blacksquare$$

If we let $k \to \infty$ in Lemma 3.4, we approach the unrestrained opportunistic policy, and deduce that

$$F \text{ is IFR} \Rightarrow \bar{F}(t) \leq \bar{F}(a) \cdot \bar{F}(t - a) \quad \text{for all } t \geq a > 0 .$$

We are now ready to establish that given F is IFR, the times between successive in-service failures decreases stochastically as the policy parameter 'a' increases. This also implies a reduction in the mean time between in-service failures.

<u>Theorem 3.5</u>:

If Unit 1 has an IFR distribution, F , then X_i , $i = 1,2, \ldots$ decreases stochastically in policy parameter $a > 0$ for all exponential G associated with Unit 2.

<u>Proof</u>:

It suffices to show that $\frac{d}{da}\bar{F}(t,a) \leq 0$ for all $a > 0$ when $t < a$, $\frac{d}{da}\bar{F}(t,a) = \frac{d}{da}\bar{F}(t) = 0$ and the theorem is trivially true.

Assume that $\frac{d}{da}\bar{F}(t,a) \leq 0$ for $t < na$. Then, for $t < (n+1)a$,

$$\frac{d}{da}\bar{F}(t,a) = \frac{d}{da}\left\{\bar{F}(t)\bar{G}(t-a) + \int_a^t \bar{F}(y)\bar{F}(t-y,a)\,dG(y-a)\right\}$$

(where $y = x + a$)

$$= \frac{1}{\mu}\bar{F}(t)\bar{G}(t-a) + \frac{1}{\mu}\int_a^t \bar{F}(y)\bar{F}(t-y,a)\,dG(y-a)$$

$$- \frac{1}{\mu}\bar{F}(a)\cdot\bar{F}(t-a,a) + \int_a^t \bar{F}(y)\cdot\left(\frac{d}{da}\bar{F}(t-y,a)\right)dG(y-a)$$

$$= \frac{1}{\mu}\{F(t,a) - \bar{F}(a)\cdot\bar{F}(t-a,a)\} +$$

$$\int_a^t \bar{F}(y)\cdot\left(\frac{d}{da}\bar{F}(t-y,a)\right)dG(y-a) \;.$$

The first term on the right hand side of Equation (19) is non-positive by Lemma 3.4. For the second term, note that when $t < (n+1)a$, $t - y \leq na$ for $y \in (a,t)$. Therefore,

$\frac{d}{da}\bar{F}(t - y,a) \leq 0$ by the induction hypothesis. Hence,

$$\frac{d}{da}\bar{F}(t,a) \leq 0 \qquad \text{for} \quad t < (n + 1)a$$

and by induction it follows that this result extends to any t when $a > 0$. ■

From Theorem 3.5 we see that when in-service failures have to be minimized, the policy parameter 'a' should be set to zero. This would mean that every time an opportunity arises (i.e. the repairman visits or Unit 2 fails) we should replace Unit 1. This would naturally entail an increase in the total number of replacements required to maintain the unit. In general, we would like to weigh the costs of in-service failure replacements against opportunistic replacements and select the optimal policy as defined in Section 3.3.

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